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Reflection Coefficients for the Generalized Jacobi Weight Functions

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In the present paper we study generalized Jacobi weight functions on the unit circle—the simplest weight functions with a finite number of algebraic singularities—and asymptotic behavior of the reflection coefficients associated with them. The real line analogues concerning the recurrence coefficients associated with generalized Jacobi weight functions in the interval are obtained.

1. POLYNOMIALS ON THE UNIT CIRCLE

Let $w(\theta)$ be a weight function on the interval $[-\pi, \pi)$ and let $\{\Phi_n(z)\}_{n=0}^{\infty}$ denote the monic orthogonal polynomials, associated with this weight function:

$$(1/2\pi)\int_{-\pi}^{\pi} \Phi_n(\zeta)\,\overline{\Phi_m(\zeta)}\,w(\theta)\,d\theta = 0, \qquad n \neq m,\ \zeta = e^{i\theta},\ \Phi_n(z) = z^n + \cdots. \tag{1}$$

In the present paper we investigate the behavior of the reflection coefficients $\Phi_n(0)$, corresponding to the generalized Jacobi weight functions (GJWF) $w(\theta)$:

$$w(\theta) = h(\theta)\prod_{v=1}^{N} |\zeta - \zeta_v|^{2\gamma_v}, \qquad \zeta = \exp(i\theta),\ \zeta_v = \exp(i\theta_v), \tag{2}$$

where

$$-\pi \leqslant \theta_N < \theta_{N-1} < \cdots < \theta_1 < \pi; \qquad 2\gamma_v > -1,\ \gamma_v \neq 0. \tag{3}$$

Regarding to the regular factor $h(\theta)$, we shall assume that it is positive differentiable function and for its derivative $h'(\theta)$ the relation

$$\omega(\delta, h') = O(1/|\ln \delta|); \qquad \delta \to 0 \tag{4}$$

holds, where $\omega(\delta, f)$ is modulus of continuity of the function $f(\theta)$.

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The following assertion is the main result of our paper.

THEOREM 1. *For the* GJWF $w(\theta)$ (2)–(4)

$$0 < \limsup_{n \to \infty} n\, |\Phi_n(0)| < \infty \tag{5}$$

holds.

In what follows $C_1, C_2, \ldots$ denote positive constants depending on weight function only. We write $\|\cdot\|_p$, $1 \leqslant p < \infty$ for the norm in L^p space on the unit circle ($\|\cdot\|_\infty$ for sup-norm).

For the weight functions $w(\theta)$, belonging to Szegő class, i.e., $\ln w(\theta) \in L^1$, the principal tool is the Szegő function

$$D(w, z) = \exp\left\{(1/4\pi) \int_{-\pi}^{\pi} (\zeta + z)(\zeta - z)^{-1} \ln w(\theta)\, d\theta\right\}, \qquad \zeta = \exp(i\theta).$$

For the GJWF $w(\theta)$ (2) we have (cf. [1, Chap. 5, Sect. 3])

$$D(w, z) = D(h, z) \prod_{\nu=1}^{N} (1 - z\zeta_\nu^{-1})^{\gamma_\nu}, \qquad \zeta_\nu = \exp(i\theta_\nu). \tag{6}$$

We start from the following statement.

LEMMA 1. *For the function* $D(z) = D(w, z)$ (6) *there exists a polynomial* $T_n(z)$ *of degree at most n such that*

(i) $\|1 - D(\zeta)\, T_n(\zeta)\|_1 = O(1/n)$, $n \to \infty$;

(ii) $\|D(\zeta)\, T_n(\zeta)\|_\infty = O(1)$, $n \to \infty$, $\zeta = e^{i\theta}$.

Proof. To prove Lemma 1 we proceed in several steps.

Step 1. If the functions $D_1(z)$ and $D_2(z)$ satisfy (i)–(ii), then so does their product $D(z) = D_1(z)\, D_2(z)$. Indeed, let polynomials $T_{n,j}(z)$ correspond to the functions $D_j(z)$, $j = 1, 2$. Put $l = [n/2]$ and $T_n(z) = T_{l,1}(z)\, T_{l,2}(z)$. Then

$$\|1 - D(\zeta)\, T_n(\zeta)\|_1 \leqslant \|1 - D_1(\zeta)\, T_{l,1}(\zeta)\|_1 + \|D_1(\zeta)\, T_{l,1}(\zeta)\|_\infty\, \|1 - D_2(\zeta)\, T_{l,2}(\zeta)\|_1 = O(1/n);$$

$$\|D(\zeta)\, T_n(\zeta)\|_\infty = \|D_1(\zeta)\, T_{l,1}(\zeta)\, D_2(\zeta)\, T_{l,2}(\zeta)\|_\infty = O(1), \qquad n \to \infty.$$

Thus, it is sufficient to ensure the correctness of relations (i)–(ii) for every factor in (6).

Step 2. Let $D(z) = 1 - z$ and

$$Q_m(z) = m^{-1} \sum_{k=1}^{m} z^k$$

so that $\|Q_m\|_\infty = Q_m(1) = 1$, $\|Q_m\|_2^2 = m^{-1}$. Put

$$\hat{T}_n(z) = (1-z)^{-1}(1-Q_l^2(z)), \qquad l = [n/3].$$

In this case

$$\begin{aligned} \|1-(1-\zeta)\hat{T}_n(\zeta)\|_1 &= \|Q_l(\zeta)\|_2^2 = l^{-1} = O(1/n) \\ \|(1-\zeta)\hat{T}_n(\zeta)\|_\infty &= \|1-Q_l^2(\zeta)\|_\infty \leqslant 2. \end{aligned} \tag{7}$$

Let us note an identity

$$\hat{T}_n(z) = (1+Q_l(z))\frac{1-Q_l(z)}{1-z} = (1+Q_l(z))\sum_{k=0}^{l-1}(1-k/l)z^k,$$

from which it follows that

$$\|\hat{T}_n(\zeta)\|_\infty \leqslant l+1 \leqslant n. \tag{8}$$

Step 3. Let $D(z) = (1-z)^\alpha$, $0<\alpha<1$. Since the function $f(z) = (1-z)^{2-\alpha}$ is analytic in the unit disk $\mathbb{D} = \{|z|<1\}$ and continuous in the closed unit disk, then there exists a polynomial $P_m(z)$ of degree at most m such that

$$\|(1-\zeta)^{2-\alpha} - P_m(\zeta)\|_\infty \leqslant C_1 m^{\alpha-2}, \tag{9}$$

C_1 is an absolute constant (cf. [2, Chap. 3]). Put now

$$T_n(z) = P_l(z)(\hat{T}_l(z))^2, \qquad l = [n/3],$$

so that

$$\begin{aligned} \|1-(1-\zeta)^\alpha T_n(\zeta)\|_1 \leqslant{}& \|1-((1-\zeta)\hat{T}_l(\zeta))^2\|_1 \\ &+ \|((1-\zeta)\hat{T}_l(\zeta))^2(1-(1-\zeta)^{\alpha-2}P_l(\zeta))\|_1 = I_1 + I_2. \end{aligned}$$

The first term I_1 is $O(1/n)$ by virtue of Step 2. For the second term I_2 we have by (9)

$$\begin{aligned} I_2 &= \|(1-\zeta)^\alpha(\hat{T}_l(\zeta))^2((1-\zeta)^{2-\alpha} - P_l(\zeta))\|_1 \\ &\leqslant C_2 n^{\alpha-2}\left\{\int_{|\theta|\leqslant n^{-1}} |(1-\zeta)^\alpha(\hat{T}_l(\zeta))^2|\,d\theta \right. \\ &\qquad \left. + \int_{n^{-1}<|\theta|\leqslant\pi} |(1-\zeta)^\alpha(\hat{T}_l(\zeta))^2|\,d\theta\right\} \\ &= C_2 n^{\alpha-2}\{I_{21}+I_{22}\}. \end{aligned}$$

In view of (8),

$$I_{21} \leqslant n^2 \int_{|\theta| \leqslant n^{-1}} |1-\zeta|^\alpha \, d\theta \leqslant 2n^{1-\alpha}.$$

For I_{22} we have with regard to (7),

$$I_{22} = \int_{n^{-1} < |\theta| \leqslant \pi} |(1-\zeta)^{\alpha-2}| \, |(1-\zeta) \hat{T}_l(\zeta)|^2 \, d\theta$$

$$\leqslant 4 \int_{n^{-1} < |\theta| \leqslant \pi} |(1-\zeta)^{\alpha-2}| \, d\theta \leqslant C_3 n^{1-\alpha}.$$

Thus $I_2 = O(1/n)$ and so (i) is proved. To prove (ii) we note that, as follows from (9),

$$|(1-\zeta)^2 - (1-\zeta)^\alpha P_l(\zeta)| \leqslant C_1 \, |1-\zeta|^\alpha n^{\alpha-2}, \tag{10}$$

$$|1 - (1-\zeta)^{\alpha-2} P_l(\zeta)| \leqslant C_1 \, |1-\zeta|^{\alpha-2} n^{\alpha-2}. \tag{11}$$

Therefore for $|\theta| \leqslant n^{-1}$ we obtain, using (8) and (10),

$$\sup_{|\theta| \leqslant n^{-1}} |(1-\zeta)^\alpha T_n(\zeta)| = \sup_{|\theta| \leqslant n^{-1}} |(1-\zeta)^\alpha P_l(\zeta)(\hat{T}_l(\zeta))^2| = O(1), \qquad n \to \infty.$$

For $n^{-1} < |\theta| \leqslant \pi$ we have with regard to (7) and (11)

$$\sup_{n^{-1} < |\theta| \leqslant \pi} |(1-\zeta)^\alpha T_n(\zeta)|$$

$$= \sup_{n^{-1} < |\theta| \leqslant \pi} |(1-\zeta) \hat{T}_l(\zeta)|^2 \, |(1-\zeta)^{\alpha-2} P_l(\zeta)| = O(1),$$

when $n \to \infty$, which proves (ii) in the present case.

Step 4. From the considerations of Steps 1–3 it follows that the relations (i)–(ii) are valid for the functions $D(z) = (1-z)^\alpha$, $\alpha > 0$. Let now $m \geqslant 1$ be a positive integer (for our purposes it is sufficient to consider $m = 1$). For $D(z) = (1-z)^{-m}$ polynomial $T_n(z) = (1-z)^m$ obviously satisfies (i)–(ii). Hence relations (i)–(ii) are valid for $D(z) = (1-z)^\alpha$ for any real α.

Step 5. Let $D(z) = D(h, z)$. We denote by $\varphi_n(z) = \kappa_n \Phi_n(z)$, $\kappa_n > 0$ the system of orthonormal polynomials, associated with $w(\theta)$,

$$(1/2\pi) \int_{-\pi}^{\pi} \varphi_n(\zeta) \overline{\varphi_m(\zeta)} \, w(\theta) \, d\theta = \delta_{n,m}, \tag{12}$$

and let $\varphi_n^*(z) = z^n \overline{\varphi_n(1/\bar{z})}$ be the reverse polynomials. As is known (cf. [3, Thm. 1.2]), for the regular weight functions $w(\theta) = h(\theta)$ (in particular, for $h(\theta)$, satisfying (4)), the following estimate holds:

$$\|D^{-1}(h, \zeta) - \varphi_n^*(\zeta)\|_\infty \leqslant C_4 \ln n n^{-1} \omega(1/n, h').$$

Under hypotheses (4) this implies (i)–(ii) for $D(z) = D(h, z)$. Hence the assertion of Lemma 1 is verified.

Proof of Theorem 1. We use the following expression for the reflection coefficients (cf. [4, formula (4)]):

$$\begin{aligned}\overline{\Phi_{n+1}(0)} &= (\kappa/2\pi) \int_{-\pi}^{\pi} D^{-1}(w, \zeta)\, \overline{\Phi_{n+1}(\zeta)}\, w(\theta)\, d\theta \\ &= (\kappa/2\pi) \int_{-\pi}^{\pi} (D^{-1}(w, \zeta) - T_n(\zeta))\, \overline{\Phi_{n+1}(\zeta)} \\ &\quad \times w(\theta)\, d\theta, \qquad \kappa = \lim_{n\to\infty} \kappa_n.\end{aligned}$$

Here $T_n(z)$ is an arbitrary polynomial of degree at most n. Therefore

$$|\Phi_{n+1}(0)| \leqslant C_5 \int_{-\pi}^{\pi} |1 - D(w, \zeta)\, T_n(\zeta)|\, |D^{-1}(w, \zeta)\, \Phi_{n+1}(\zeta)|\, w(\theta)\, d\theta. \tag{13}$$

It is known (cf. [5, p. 29; 6]), that the orthonormal polynomials $\varphi_n(z)$, corresponding to the GJWF $w(\theta)$ (2)–(4) admit the estimate on the unit circle

$$\begin{aligned}&C_6 \prod_{\nu=1}^{N} \{|\zeta - \zeta_\nu| + (n+1)^{-1}\}^{-\gamma_\nu} \\ &\quad\leqslant |\varphi_{n+1}(\zeta)| \leqslant C_7 \prod_{\nu=1}^{N} \{|\zeta - \zeta_\nu| + (n+1)^{-1}\}^{-\gamma_\nu}, \qquad \zeta_\nu = \exp(i\theta_\nu).\end{aligned} \tag{14}$$

The same estimate is obviously true for the monic orthogonal polynomials $\Phi_n(z)$ as well, since within the Szegő class

$$0 < \kappa_0 \leqslant \kappa_n < \kappa < \infty.$$

Using the explicit form of the Szegő function $D(w, z)$ (6) and the weight function $w(\theta)$ (2), we obtain

$$\begin{aligned}|D^{-1}(w, \zeta)\, \Phi_{n+1}(\zeta)|\, w(\theta) &\leqslant C_8 \prod_{\nu=1}^{N} \left\{\frac{|\zeta - \zeta_\nu|}{|\zeta - \zeta_\nu| + (n+1)^{-1}}\right\}^{\gamma_\nu} \\ &\leqslant C_8 \prod_{\nu \in J} \left\{\frac{|\zeta - \zeta_\nu|}{|\zeta - \zeta_\nu| + (n+1)^{-1}}\right\}^{\gamma_\nu},\end{aligned} \tag{15}$$

where $J=\{\nu\colon 1\leqslant \nu\leqslant N, \gamma_\nu<0\}$. Let $a=(1/2)\min_k |\theta_{k+1}-\theta_k|$. Consider the sets

$$\Gamma_j=\{\theta\colon |\theta-\theta_j|\leqslant an^{-1}\},\qquad \Gamma=\bigcup_{j\in J}\Gamma_j,\quad \hat{\Gamma}=[\pi,\pi)\backslash\Gamma.$$

We have (see (13), (15))

$$\begin{aligned}|\Phi_{n+1}(0)|&\leqslant C_9\sum_{j\in J}\int_{\Gamma_j}|1-D(w,\zeta)\,T_n(\zeta)|\prod_{\nu\in J}\left\{\frac{|\zeta-\zeta_\nu|+(n+1)^{-1}}{|\zeta-\zeta_\nu|}\right\}^{|\gamma_\nu|}d\theta\\&\quad+C_9\int_{\hat{\Gamma}}|1-D(w,\zeta)\,T_n(\zeta)|\prod_{\nu\in J}\left\{\frac{|\zeta-\zeta_\nu|+(n+1)^{-1}}{|\zeta-\zeta_\nu|}\right\}^{|\gamma_\nu|}d\theta\\&=\sum_{j\in J}I_j+\hat{I}.\end{aligned}$$

Let the polynomial $T_n(z)$ in the previous relation correspond to the function $D(w,z)$ by Lemma 1. On the set $\hat{\Gamma}$ we have $|\theta-\theta_\nu|>an^{-1}$ for every $\nu\in J$, so that

$$\prod_{\nu\in J}\left\{\frac{|\zeta-\zeta_\nu|+(n+1)^{-1}}{|\zeta-\zeta_\nu|}\right\}^{|\gamma_\nu|}\leqslant C_{10},$$

and by means of Lemma 1(i)

$$\hat{I}\leqslant C_{11}\,\|1-D(w,\zeta)\,T_n(\zeta)\|_1=O(1/n).$$

If $\theta\in\Gamma_j$, then by Lemma 1(ii) we have

$$\begin{aligned}I_j&\leqslant C_{12}\int_{\Gamma_j}|1-D(w,\zeta)\,T_n(\zeta)|\left\{\frac{|\zeta-\zeta_j|+(n+1)^{-1}}{|\zeta-\zeta_j|}\right\}^{|\gamma_j|}d\theta\\&\leqslant C_{12}\,\|1-D(w,\zeta)\,T_n(\zeta)\|_\infty\int_{\Gamma_j}\left\{\frac{|e^{i(\theta-\theta_j)}-1|+(n+1)^{-1}}{|e^{i(\theta-\theta_j)}-1|}\right\}^{|\gamma_j|}d\theta\\&\leqslant C_{13}n^{-|\gamma_j|}\int_{\Gamma_j}|\sin(\theta-\theta_j)/2|^{-|\gamma_j|}\,d\theta\\&=O(1/n),\qquad n\to\infty,\end{aligned}\tag{16}$$

and the right-hand inequality (5) is proved.

Proceeding to the proof of the left-hand inequality (5), we consider the value (cf. [7, Chap. 2]):

$$\delta_n(w)=\|1-\kappa_n\kappa^{-1}\varphi_n^*(\zeta)\,D(w,\zeta)\|_2=\kappa^{-1}\left\{\sum_{j=n+1}^{\infty}|\kappa_j\Phi_j(0)|^2\right\}^{1/2}.\tag{17}$$

It easily follows from the right-hand inequality (5) (cf. [8, Thm. 4]) that now

$$\delta_n(w) = O(n^{-1/2}), \qquad n \to \infty.$$

We will now show that

$$\liminf_{n \to \infty} n^{1/2}\, \delta_n(w) > 0. \tag{18}$$

It is convenient to distinguish the following two cases.

1. Assume, that $\gamma_1 < 0$. Let $\Delta = \{\theta : |\theta - \theta_1| \leqslant \varepsilon n^{-1}\}$, where a sufficiently small $\varepsilon > 0$ will be chosen later on. By the triangle inequality

$$\begin{aligned}(2\pi)^{1/2}\, \delta_n(w) &\geqslant \left\{\int_\Delta |1 - \kappa_n \kappa^{-1} \varphi_n^*(\zeta)\, D(w, \zeta)|^2\, d\theta\right\}^{1/2} \\ &\geqslant \left\{\int_\Delta |\kappa_n \kappa^{-1} \varphi_n^*(\zeta)\, D(w, \zeta)|^2\, d\theta\right\}^{1/2} - (2\varepsilon/n)^{1/2}.\end{aligned}$$

Using the left-hand inequality (14) and the explicit form of the Szegő function (6), we obtain, as above in (16),

$$\begin{aligned}\int_\Delta |\kappa_n \kappa^{-1} \varphi_n^*(\zeta)\, D(w, \zeta)|^2\, d\theta &\geqslant C_{14} \int_\Delta \left\{\frac{|\zeta - \zeta_1| + n^{-1}}{|\zeta - \zeta_1|}\right\}^{2|\gamma_1|} d\theta \\ &\geqslant C_{15} n^{-2|\gamma_1|} \int_{-\varepsilon n^{-1}}^{\varepsilon n^{-1}} |\theta|^{-2|\gamma_1|}\, d\theta \\ &= C_{16} \varepsilon n^{-1} \varepsilon^{-2|\gamma_1|}.\end{aligned}$$

The constant C_{16} here depends on the weight function $w(\theta)$ only (but does not depend on the choice of ε). We can choose now the constant ε from the condition $C_{16} \varepsilon^{-2|\gamma_1|} > 3$, which entails

$$(2\pi)^{1/2}\, \delta_n(w) \geqslant (3^{1/2} - 2^{1/2})(\varepsilon n^{-1})^{1/2},$$

and so does the inequality (18).

2. Assume that $\gamma_1 > 0$. By repeating the argument above, we obtain

$$(2\pi)^{1/2}\, \delta_n(w) \geqslant (2\varepsilon/n)^{1/2} - \left\{\int_\Delta |\kappa_n \kappa^{-1} \varphi_n^*(\zeta)\, D(w, \zeta)|^2\, d\theta\right\}^{1/2},$$

and as follows from the right-hand inequality (14),

$$\begin{aligned}\int_\Delta |\kappa_n \kappa^{-1} \varphi_n^*(\zeta)\, D(w, \zeta)|^2\, d\theta &\leqslant C_{17} \int_\Delta \left\{\frac{|\zeta - \zeta_1|}{|\zeta - \zeta_1| + n^{-1}}\right\}^{2\gamma_1} d\theta \\ &\leqslant C_{18} n^{2\gamma_1} \int_{-\varepsilon n^{-1}}^{\varepsilon n^{-1}} |\theta|^{2\gamma_1}\, d\theta \\ &\leqslant C_{19} \varepsilon n^{-1} \varepsilon^{2\gamma_1}.\end{aligned}$$

Choosing the constant ε from the condition $C_{19}\varepsilon^{2\gamma_1} < 1$, we arrive at the inequality

$$(2\pi)^{1/2}\,\delta_n(w) \geqslant (2^{1/2}-1)(\varepsilon n^{-1})^{1/2},$$

which proves (18).

Assume, on the contrary, that

$$\limsup_{n\to\infty} n\,|\Phi_n(0)| = 0,$$

i.e., $\lim_{n\to\infty} n\,|\Phi_n(0)| = 0$, and hence $n\,|\Phi_n(0)| \leqslant b$, $n \geqslant n_1(b)$ for any

$$0 < b < \liminf_{n\to\infty} n^{1/2}\,\delta_n(w). \tag{19}$$

But then (see (17))

$$\delta_n(w) \leqslant \left\{ \sum_{j=n+1}^{\infty} |\Phi_j(0)|^2 \right\}^{1/2} \leqslant bn^{-1/2}, \qquad n^{1/2}\,\delta_n(w) \leqslant b;$$

for $n \geqslant n_1(b)$. The latter relation contradicts (19) and so the left-hand inequality (5) is established.

Theorem 1 is now completely proved.

EXAMPLE. We consider the Jacobi weight function $w(\theta)$ on the unit circle,

$$w(\theta) = |e^{i\theta}-1|^{2\gamma_1}\,|e^{i\theta}+1|^{2\gamma_2} = 4^{\gamma_1+\gamma_2}\,|\sin\theta/2|^{2\gamma_1}\,|\cos\theta/2|^{2\gamma_2}, \tag{20}$$

and the corresponding orthogonal polynomials $\Phi_n(z)$. An explicit expression for the reflection coefficients in this case has been obtained in [8, Sect. 2] (in formula (2.8) there, one has to replace n by $n+1$):

$$\Phi_n(0) = (\gamma_1 + (-1)^n\,\gamma_2)(n+\gamma_1+\gamma_2)^{-1}.$$

Let $k \geqslant 2$ be a positive integer. We introduce a GJWF $w_k(\theta) = w(k\theta)$. It is actually not hard to see that corresponding orthogonal polynomials $\Phi_{k,n}(z)$ admit representation (cf. [9]):

$$\Phi_{k,\,kn+\nu}(z) = z^\nu \Phi_n(z^k), \qquad n = 0, 1, ..., \quad \nu = 0, 1, ..., k-1.$$

Hence

$$\Phi_{k,m}(0) = \begin{cases} \Phi_{m/k}(0), & m \equiv 0 \pmod{k}, \\ 0, & m \not\equiv 0 \pmod{k}, \end{cases}$$

so that

$$0=\liminf_{n\to\infty} n\,|\Phi_{k,n}(0)|<\limsup_{n\to\infty} n\,|\Phi_{k,n}(0)|=|\gamma_1|+|\gamma_2|.$$

2. Polynomials on the Real Line

We are now able to establish an analogue of Theorem 1 for the interval $[-1,1]$.

THEOREM 2. *Let $f(x)$ be a* GJWF *on the interval* $[-1,1]$,

$$f(x)=g(x)\prod_{\nu=1}^{N}|x-x_\nu|^{2\alpha_\nu}, \tag{21}$$

where

$$-1\leqslant x_N<x_{N-1}<\cdots<x_1\leqslant 1, \qquad 2\alpha_\nu>-1,\ \alpha_\nu\neq 0,$$

and $g(x)$ is a positive differentiable function such that

$$\omega(\delta, g')=O(1/|\ln\delta|), \qquad \delta\to 0.$$

Let $p_n(x)$ be a system of orthonormal polynomials corresponding to the weight function $f(x)$ (21) *and satisfying the recurrence formula*

$$xp_n(x)=a_{n+1}p_{n+1}(x)+b_np_n(x)+a_np_{n-1}(x).$$

Then

$$a_n=1/2+O(1/n), \qquad b_n=O(1/n), \qquad n\to\infty.$$

Proof. We can transfer to the unit circle using the standard transformation $w(\theta)=f(\cos\theta)\,|\sin\theta|$. It can be readily shown that the GJWF $w(\theta)$ satisfies the hypotheses of Theorem 1. Corresponding reflection coefficients $\Phi_n(0)$ and the coefficients a_n, b_n are related by means of the following formulas, which have actually been found by Ya. L. Geronimus [10, Sect. 31] (see also [11, formula (5.10)]):

$$4a_n^2=(1+\Phi_{2n-2}(0))(1-\Phi_{2n-1}^2(0))(1-\Phi_{2n}(0)),$$
$$2b_n=\Phi_{2n-1}(0)(1-\Phi_{2n}(0))-\Phi_{2n+1}(0)(1+\Phi_{2n}(0)).$$

The rest is immediate from Theorem 1.

By using the main Theorem 1 we are able to investigate the asymptotics of the second type of polynomials corresponding to the GJWF (2)–(4) and the smoothness properties (absolute continuity and the absence of mass points) of the appropriate measure. These and some other problems regarding the second type of polynomials will be treated elsewhere.

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